

(PARA)-KÄHLER WEYL STRUCTURES

P. GILKEY AND S. NIKČEVIĆ

ABSTRACT. We work in both the complex and in the para-complex categories and examine (para)-Kähler Weyl structures in both the geometric and in the algebraic settings. The higher dimensional setting is quite restrictive. We show that any (para)-Kähler Weyl algebraic curvature tensor is in fact Riemannian in dimension $m \geq 6$; this yields as a geometric consequence that any (para)-Kähler Weyl geometric structure is trivial for $m \geq 6$. By contrast, the 4-dimensional setting is, as always, rather special as it turns out that there are (para)-Kähler Weyl algebraic curvature tensors which are not Riemannian if $m = 4$. Since every (para)-Kähler Weyl algebraic curvature tensor is geometrically realizable and since every 4-dimensional Hermitian manifold admits a unique (para)-Kähler Weyl structure, there are also non-trivial 4-dimensional Hermitian (para)-Kähler Weyl manifolds.

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1. INTRODUCTION

Let ∇ be a torsion free connection on a pseudo-Riemannian manifold (M, g) of even dimension $m = 2\bar{m} \geq 4$. The triple (M, g, ∇) is said to be a *Weyl structure* if there exists a smooth 1-form ϕ so that $\nabla g = -2\phi \otimes g$. Such a geometric structure was introduced by Weyl [37] in an attempt to unify gravity with electromagnetism. Although this approach failed for physical reasons, these geometries are still studied for their intrinsic interest [2, 10, 21, 27, 28]; they also appear in the mathematical physics literature [12, 20, 26]. Weyl geometry is relevant to submanifold geometry [25] and to contact geometry [15]. The pseudo-Riemannian setting also is important [1, 24, 32] as are para-complex geometries [11, 13]. See also [9, 22, 30, 31] for related results. The literature in the field is vast and we can only give a flavor of it for reasons of brevity. We shall be primarily interested in the Hermitian setting. However since there are applications to higher signature geometry, we include the pseudo-Hermitian context as well; similarly we treat para-Hermitian geometries as they can be studied with little additional effort.

Section 1.1 of the Introduction deals with the real setting. In Theorem 1.1, we recall the basic theorems of geometric realizability for affine, Riemannian, and Weyl curvature models and in Theorem 1.2 provide various characterizations of the notion of a trivial Weyl structure. Section 1.2 treats the (para)-Kähler setting. In Theorem 1.3 we recall geometric realizability results for (para)-Kähler affine and (para)-Kähler Riemannian curvature models. Theorem 1.4 presents results in the geometric setting for (para)-Kähler Weyl manifolds. Theorem 1.5 is one of the two main results of this paper: every (para)-Kähler curvature model is geometrically realizable. The proof of Theorem 1.5 relies on a curvature decomposition result; the second main result of the paper, Theorem 1.6, discusses the space of (para)-Kähler Weyl algebraic curvature tensors.

1.1. Riemannian, Affine, and Weyl geometry. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space of signature (p, q) and dimension $m = p + q$; an inner product of signature $(0, 4)$ is positive definite. A 4-tensor $A \in \otimes^4 V^*$ is said to be a *Riemannian algebraic curvature tensor* if A satisfies the symmetries of the Riemann curvature tensor,

namely:

$$A(x, y, z, w) + A(y, x, z, w) = 0, \quad (1.a)$$

$$A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) = 0, \quad (1.b)$$

$$A(x, y, z, w) = A(w, z, x, y). \quad (1.c)$$

Let $\mathfrak{R}(V)$ be the subspace of $\otimes^4 V^*$ which consists of all tensors satisfying these relations. We say that a triple $\mathcal{R} := (V, \langle \cdot, \cdot \rangle, A)$ is a *Riemannian curvature model* if $A \in \mathfrak{R}(V)$. One says that \mathcal{R} is *geometrically realizable* by a pseudo-Riemannian manifold if there is a point P of some pseudo-Riemannian manifold (M, g) and if there is an isomorphism $\Phi : V \rightarrow T_P M$ so:

$$\Phi^* g_P = \langle \cdot, \cdot \rangle \quad \text{and} \quad \Phi^* R_P^g = A$$

where R^g is the curvature tensor of the Levi-Civita connection ∇^g on M .

Affine differential geometry extends Riemannian geometry. A pair (M, ∇) is said to be an *affine manifold* if ∇ is a torsion free connection on the tangent bundle TM . The curvature R^∇ of the connection ∇ then satisfies the identities of Equations (1.a) and (1.b) but need no longer satisfy Equation (1.c); if $A \in \otimes^4 V^*$, one says A is an *affine algebraic curvature tensor* if A satisfies Equations (1.a) and (1.b) and one lets $\mathfrak{A}(V)$ be the set of all such tensors. Note that the corresponding *curvature operator* \hat{A} and the curvature tensor A are related by the identity

$$\langle \hat{A}(x, y)z, w \rangle = A(x, y, z, w).$$

The pair $\mathcal{A} := (V, A)$ is said to be an *affine curvature model* if $A \in \mathfrak{A}(V)$; such an \mathcal{A} is said to be *geometrically realizable by an affine manifold* if there is a point P of some affine manifold (M, ∇) and if there is an isomorphism $\Phi : V \rightarrow T_P M$ so that $\Phi^* R_P^\nabla = A$.

Weyl geometry is in a sense midway between Riemannian geometry and affine geometry. A triple (M, g, ∇) is said to be a *Weyl manifold* if (M, g) is a pseudo-Riemannian manifold, if (M, ∇) is an affine manifold, and if there exists a smooth 1-form ϕ on M so that the structures are related by the equation:

$$\nabla g = -2\phi \otimes g. \quad (1.d)$$

Define the *Ricci-tensor* $\rho = \rho_\nabla$ and the *alternating Ricci tensor* $\rho_a = \rho_{a, \nabla}$ by:

$$\begin{aligned} \rho(x, y) &:= \text{Tr}(z \rightarrow R(z, x)y), \\ \rho_a(x, y) &:= \frac{1}{2}\{\rho(x, y) - \rho(y, x)\}. \end{aligned}$$

There is an additional curvature symmetry which pertains in Weyl geometry (see, for example, the discussion in [17]):

$$R(x, y, z, w) + R(x, y, w, z) = -\frac{4}{m}\rho_a(x, y)g(z, w). \quad (1.e)$$

The defining 1-form ϕ is related to the curvature by the equation:

$$d\phi = -\frac{1}{m}\rho_a. \quad (1.f)$$

Let $\mathfrak{W}(V) \subset \otimes^4(V^*)$ be space of 4-tensors satisfying Equations (1.a), (1.b), and (1.e); these are the Weyl algebraic curvature tensors. If $A \in \mathfrak{R}$, then $\rho_a = 0$ and $A(x, y, z, w) + A(x, y, w, z) = 0$. Consequently:

$$\mathfrak{R}(V) \subset \mathfrak{W}(V) \subset \mathfrak{A}(V).$$

A triple $\mathcal{W} := (V, \langle \cdot, \cdot \rangle, A)$ is said to be a *Weyl curvature model* if $A \in \mathfrak{W}(V)$. The notion of geometric realizability is defined analogously in this setting.

We refer to [16, 8, 17] for the proof of the following result; the first two assertions are, of course, well known:

Theorem 1.1.

- (1) *Every Riemannian curvature model is geometrically realizable by a pseudo-Riemannian manifold.*
- (2) *Every affine curvature model is geometrically realizable by an affine manifold.*
- (3) *Every Weyl curvature model is geometrically realizable by a Weyl manifold.*

Weyl geometry is a conformal theory; if $g_1 = e^{2f}g$ is conformally equivalent to g and if (M, g, ∇) is a Weyl manifold, then (M, g_1, ∇) is again a Weyl manifold with associated 1-form ϕ_1 given by $\phi_1 = \phi - df$. One has the following well known result characterizing trivial Weyl structures (see, for example, [17]):

Theorem 1.2. *Let (M, g, ∇) be a Weyl manifold with $H^1(M; \mathbb{R}) = 0$. The following assertions are equivalent and if any is satisfied, then the Weyl structure is said to be trivial.*

- (1) $d\phi = 0$.
- (2) $\nabla = \nabla^{g_1}$ for some conformally equivalent metric g_1 .
- (3) $R^\nabla \in \mathfrak{A}$.

1.2. Kähler geometry. We now pass from the real to the (para)-complex setting. Let V be a real vector space of even dimension $m = 2\bar{m}$. A complex structure on V is an endomorphism J_- of V so $J_-^2 = -\text{Id}$. Similarly, a para-complex structure on V is an endomorphism J_+ of V so $J_+^2 = \text{Id}$ and $\text{Tr}(J_+) = 0$; this trace-free condition is automatic in the complex setting but must be imposed in the para-complex setting. It is convenient to introduce the notation J_\pm in order to have a common formulation in both contexts although we shall never be considering both structures simultaneously. In the geometric setting, (M, J_\pm) is said to be an *almost (para)-complex manifold* if J_\pm is a smooth endomorphism of the tangent bundle so that $(T_P M, J_\pm)$ is a (para)-complex structure for every $P \in M$. The almost (para)-complex structure J_\pm is said to be *integrable* and the pair (M, J_\pm) is said to be a *(para)-complex manifold* if there are coordinate charts $(x^1, y^1, \dots, x^{\bar{m}}, y^{\bar{m}})$ covering M so that:

$$J_\pm \left\{ \frac{\partial}{\partial x_i} \right\} = \frac{\partial}{\partial y_i} \quad \text{and} \quad J_\pm \left\{ \frac{\partial}{\partial y_i} \right\} = \pm \frac{\partial}{\partial x_i} \quad \text{for } 1 \leq i \leq \bar{m}. \quad (1.g)$$

If (M, J_\pm) is an almost (para)-complex manifold and if ∇ is a torsion free connection on M , then (M, J_\pm, ∇) is said to be a *Kähler affine manifold* if $\nabla J_\pm = 0$; this assumption then implies that J_\pm is integrable. The curvature satisfies an extra symmetry in this setting:

$$R(x, y, z, w) = \mp R(x, y, J_\pm z, J_\pm w). \quad (1.h)$$

A (para)-complex pseudo-Riemannian manifold (M, g, J_\pm) is said to be a *(para)-Kähler Hermitian manifold* if $J_\pm^* g = \mp g$ and $\nabla^g J_\pm = 0$. Finally, a (para)-complex-Riemannian Weyl manifold (M, g, J_\pm, ∇) is said to be a *(para)-Kähler Weyl manifold* if $\nabla J_\pm = 0$.

We now pass to the algebraic context. Define the space of (para)-Kähler tensors \mathfrak{K}_\pm , the space of (para)-Kähler affine algebraic curvature tensors $\mathfrak{K}_{\pm, \mathfrak{A}}$, the space of (para)-Kähler Riemannian algebraic curvature tensors $\mathfrak{K}_{\pm, \mathfrak{R}}$, and the space of (para)-Kähler Weyl algebraic curvature tensors $\mathfrak{K}_{\pm, \mathfrak{W}}$ by setting, respectively:

$$\begin{aligned} \mathfrak{K}_\pm &:= \{A \in \otimes^4 V^* : A(x, y, z, w) = \mp A(x, y, J_\pm z, J_\pm w)\}, \\ \mathfrak{K}_{\pm, \mathfrak{A}} &:= \mathfrak{K}_\pm \cap \mathfrak{A}, \quad \mathfrak{K}_{\pm, \mathfrak{R}} := \mathfrak{K}_\pm \cap \mathfrak{R}, \quad \mathfrak{K}_{\pm, \mathfrak{W}} := \mathfrak{K}_\pm \cap \mathfrak{W}. \end{aligned}$$

A triple $\mathcal{K}_\mathcal{A} = (V, J_\pm, A)$ is said to be a *(para)-Kähler affine curvature model* if (V, J_\pm) is (para)-complex and if $A \in \mathfrak{K}_{\pm, \mathfrak{A}}$. A quadruple $\mathcal{K}_\mathcal{R} = (V, \langle \cdot, \cdot \rangle, J_\pm, A)$ is said to be a *(para)-Kähler Hermitian curvature model* if $J_\pm^* \langle \cdot, \cdot \rangle = \mp \langle \cdot, \cdot \rangle$ and if

$A \in \mathfrak{K}_{\pm, \mathfrak{R}}$. A quadruple $\mathcal{K}_{\mathcal{W}} = (V, \langle \cdot, \cdot \rangle, J_{\pm}, A)$ is said to be a *(para)-Kähler Weyl curvature model* if $J_{\pm}^* \langle \cdot, \cdot \rangle = \mp \langle \cdot, \cdot \rangle$ and if $A \in \mathfrak{K}_{\pm, \mathfrak{W}}$.

Let $\langle \cdot, \cdot \rangle$ have signature (p, q) ; if $p = 0$, then $\langle \cdot, \cdot \rangle$ is positive definite while if $q = 0$, then $\langle \cdot, \cdot \rangle$ is negative definite. In the para-complex setting, $p = q$ so $\langle \cdot, \cdot \rangle$ is necessarily indefinite. In the complex setting, p and q must both be even; we emphasize that we do not assume necessarily that the inner product is positive definite. We refer to [5] for the proof of Assertion (1) and to [4] for the proof of Assertion (2) in the following result:

Theorem 1.3.

- (1) *Every (para)-Kähler affine curvature model is geometrically realizable by a (para)-Kähler affine manifold.*
- (2) *Every (para)-Kähler Hermitian curvature model is geometrically realizable by a (para)-Kähler Hermitian manifold.*

The (para)-Kähler form Ω_{\pm} is defined by the identity:

$$\Omega_{\pm}(x, y) := g(x, J_{\pm}y).$$

Let δ be the co-derivative. We refer to [29, 35, 36] for the proof of Assertion (1) in the following result in the positive definite setting – the generalization to the indefinite setting is immediate. We refer to [23] for the proof of Assertion (2) in the Riemannian setting – the extension to the general setting is immediate:

Theorem 1.4.

- (1) *Let $m \geq 6$. If (M, g, J_{\pm}, ∇) is a (para)-Kähler Weyl manifold, then the associated Weyl structure is trivial, i.e. locally there is a conformally equivalent metric g_1 so that (M, g_1, J_{\pm}) is Kähler and so that $\nabla = \nabla^{g_1}$.*
- (2) *Every (para)-Hermitian manifold of dimension 4 admits a unique (para)-Kähler Weyl structure defined by taking $\phi = \pm \frac{1}{2} J_{\pm}^* \delta \Omega_{\pm}$.*

The following theorem is the first main result of this paper:

Theorem 1.5. *Every (para)-Kähler Weyl curvature model is geometrically realizable by a (para)-Kähler Weyl manifold.*

Curvature decompositions play a central role in modern differential geometry. The following theorem is the second main result of this paper and will play a central role in the proof of Theorem 1.5:

Theorem 1.6. *Let $(V, \langle \cdot, \cdot \rangle, J_{\pm})$ be a (para)-Hermitian vector space.*

- (1) *If $m \geq 6$, then $\mathfrak{K}_{\pm, \mathfrak{W}} = \mathfrak{K}_{\pm, \mathfrak{R}}$.*
- (2) *If $m = 4$, then $\mathfrak{K}_{\pm, \mathfrak{W}} = \mathfrak{K}_{\pm, \mathfrak{R}} \oplus L_{0, \mp}^2$ where*

$$\rho_a : L_{0, \mp}^2 \xrightarrow{\sim} \Lambda_{0, \mp}^2 := \{\Phi \in \Lambda^2(V^*) : \Phi \perp \Omega_{\pm} \text{ and } J_{\pm}^* \Phi = \mp \Phi\}.$$

Theorem 1.6 is one of the facts about 4-dimensional geometry that distinguishes it from the higher dimensional setting; the module $L_{0, \mp}^2$ provides additional curvature possibilities if $m = 4$.

Curvature decompositions are fundamental in establishing geometrical realizability results. For example, we can use Theorem 1.6 (1) to establish Theorem 1.4 (1) as follows. Suppose that (M, g, J_{\pm}, ∇) is a (para)-Kähler Weyl manifold of dimension $m \geq 6$. By Theorem 1.6, $R^{\nabla} \in \mathfrak{K}_{\pm, \mathfrak{R}} \subset \mathfrak{R}$. By Theorem 1.2, there is a locally conformally equivalent metric g_1 so that $\nabla = \nabla^{g_1}$; g_1 is globally defined if $H^1(M; \mathbb{R}) = 0$.

Here is a brief outline to the remainder of this paper. In Section 2, we review well known previous results concerning curvature decompositions that we shall need. Theorem 1.6 is established in Section 3 and Theorem 1.5 is established in Section 4.

2. CURVATURE DECOMPOSITIONS

In Section 2.1, the structure groups \mathcal{O} , \mathcal{U}_\pm , and \mathcal{U}_\pm^* will be defined and the fundamental facts needed from representation theory will be established. In Section 2.2, results of Singer and Thorpe [33] giving the decomposition of \mathfrak{R} and results of Higa [18, 19] giving the decomposition of \mathfrak{W} as an \mathcal{O} -module will be presented. In Section 2.3 the Tricerri–Vanhecke decomposition [34] of the space of Riemannian algebraic curvature tensors \mathfrak{R} and the space of Kähler algebraic curvature tensors $\mathfrak{R}_{\pm, \mathfrak{R}}$ as \mathcal{U}_\pm^* modules will be outlined; this will rise to the decomposition of the space of Weyl algebraic curvature tensors \mathfrak{W} as a \mathcal{U}_\pm^* module. As we shall not need the decomposition of $\mathfrak{R}_{\pm, \mathfrak{R}}$ as a \mathcal{U}_\pm^* module, we shall omit this decomposition and instead refer to the discussion in [6].

2.1. Representation theory. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The orthogonal group \mathcal{O} is the subgroup of all invertible linear transformations of V preserving the inner product. If $(V, \langle \cdot, \cdot \rangle, J_\pm)$ is a (para)-Hermitian vector space, define:

$$\begin{aligned}\mathcal{U}_\pm &:= \{T \in \mathcal{O} : TJ_\pm = J_\pm T\}, \\ \mathcal{U}_\pm^* &:= \{T \in \mathcal{O} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}.\end{aligned}$$

It is convenient to work with the \mathbb{Z}_2 extensions \mathcal{U}_\pm^* as we may then interchange the roles of J_\pm and $-J_\pm$. Let χ be the \mathbb{Z}_2 valued character of \mathcal{U}_\pm^* so that:

$$J_\pm T = \chi(T)TJ_\pm \quad \text{and} \quad T^*\Omega_\pm = \chi(T)\Omega_\pm \quad \text{for } T \in \mathcal{U}_\pm^*.$$

By an abuse of notation, we identify χ with the associated 1-dimensional module. We can extend $\langle \cdot, \cdot \rangle$ to a natural non-degenerate inner product on $\otimes^k V$ and $\otimes^k V^*$. The following observation is fundamental in the subject:

Lemma 2.1. *Let $G \in \{\mathcal{O}, \mathcal{U}_-, \mathcal{U}_-^*, \mathcal{U}_+^*\}$ and let ξ be a G -submodule of $\otimes^k V^*$. Then the restriction of the inner product on $\otimes^k V^*$ to ξ is non-degenerate.*

Proof. Let $\{e_i\}$ be an orthonormal basis for V and let $\{e^i\}$ be the associated dual basis for V^* . If $I = (i_1, \dots, i_k)$ is a multi-index, set $e^I = e^{i_1} \otimes \dots \otimes e^{i_k}$. Then:

$$(e^I, e^J) := \langle e^{i_1}, e^{j_1} \rangle \dots \langle e^{i_k}, e^{j_k} \rangle = \begin{cases} 0 & \text{if } I \neq J \\ \pm 1 & \text{if } I = J \end{cases}. \quad (2.a)$$

Let $Te_i = \langle e_i, e_i \rangle \cdot e_i$ define an element $T \in \mathcal{O}$. Suppose that ξ is an \mathcal{O} invariant subspace of $\otimes^k V^*$. Decompose $\xi = \xi_+ \oplus \xi_-$ and decompose $\otimes^k V^* = W_+ \oplus W_-$ into the ± 1 eigenspaces of T . Since $T \in \mathcal{O}$, these decompositions are orthogonal direct sums. By Equation (2.a), W_+ is spacelike and W_- is timelike. Since $\xi_\pm \subset W_\pm$, ξ_+ is spacelike and ξ_- is timelike; the Lemma now follows in this special case. If $G = \mathcal{U}_-$ or if $G = \mathcal{U}_-^*$, then we can choose the orthonormal basis so that

$$J_- e_{2\nu-1} = e_{2\nu} \quad \text{and} \quad J_- e_{2\nu} = -e_{2\nu-1}.$$

Since $J_-^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$, $J_- T = T J_-$. Thus $T \in G$ and the same argument pertains. Finally suppose $G = \mathcal{U}_+^*$. We can choose the basis so

$$J_+ e_{2\nu-1} = e_{2\nu} \quad \text{and} \quad J_+ e_{2\nu} = e_{2\nu-1}$$

where $e_{2\nu-1}$ is spacelike and $e_{2\nu}$ is timelike. We now have $T \in \mathcal{U}_+^* - \mathcal{U}_+$. \square

We note that Lemma 2.1 fails for the group $G = \mathcal{U}_+$. For example, let V_\pm be the ± 1 eigenspaces of J_+ ; then $J_\pm V_\pm = V_\pm$ and V_\pm is totally isotropic. We can combine Lemma 2.1 with same arguments as used in the positive definite setting to establish the following result; we omit details in the interests of brevity:

Lemma 2.2. *Let $G \in \{\mathcal{O}, \mathcal{U}_-, \mathcal{U}_-^*, \mathcal{U}_+^*\}$ and let ξ be a G -submodule of $\otimes^k V^*$.*

- (1) *There is an orthogonal direct sum decomposition of $\xi = \xi_1 \oplus \dots \oplus \xi_k$ into irreducible G -submodules of ξ . The multiplicity with which a given irreducible G -module η appears in ξ is independent of the particular decomposition which is chosen. If ξ_1 appears with multiplicity 1 in the decomposition of ξ and if η is any G -submodule of ξ , then either $\xi_1 \subset \eta$ or $\xi_1 \perp \eta$.*
- (2) *If $\xi_1 \rightarrow \xi \rightarrow \xi_2$ is a short exact sequence of G -modules, then ξ is isomorphic to $\xi_1 \oplus \xi_2$ as a G -module.*

We can illustrate Lemma 2.2 as follows. Decompose

$$\otimes^2 V^* = \Lambda^2(V^*) \oplus S^2(V^*)$$

as the direct sum of the alternating and the symmetric bilinear forms. We can further decompose $\Lambda^2(V^*) = \Lambda_{\pm}^2 \oplus \chi \oplus \Lambda_{0,\mp}^2$ and $S^2(V^*) = S_{\pm}^2 \oplus \mathbf{1} \oplus S_{0,\mp}^2$ where

$$\begin{aligned} \Lambda_{\pm}^2 &:= \{\omega \in \Lambda^2 : J_{\pm}^* \omega = \pm \omega\}, & \chi &:= \Omega_{\pm} \cdot \mathbb{R}, \\ \Lambda_{0,\mp}^2 &:= \{\omega \in \Lambda^2 : J_{\pm}^* \omega = \mp \omega, \omega \perp \Omega_{\pm}\}, \\ S_{\pm}^2 &:= \{\theta \in S^2 : J_{\pm}^* \theta = \pm \theta\}, & \mathbf{1} &:= \langle \cdot, \cdot \rangle \cdot \mathbb{R}, \\ S_{0,\mp}^2 &:= \{\theta \in S^2 : J_{\pm}^* \theta = \mp \theta, \theta \perp \langle \cdot, \cdot \rangle\}. \end{aligned}$$

Lemma 2.3. *Let $(V, \langle \cdot, \cdot \rangle, J_{\pm})$ be a (para)-Hermitian vector space. We have the following decomposition of $\Lambda^2(V^*)$, $S^2(V^*)$, and $\otimes^2 V^*$ into inequivalent and irreducible \mathcal{U}_{\pm}^* modules:*

$$\begin{aligned} \Lambda^2(V^*) &= \Lambda_{\pm}^2 \oplus \chi \oplus \Lambda_{0,\mp}^2, & S^2(V^*) &= S_{\pm}^2 \oplus \mathbf{1} \oplus S_{0,\mp}^2, \\ \otimes^2 V^* &= \Lambda_{\pm}^2 \oplus \chi \oplus \Lambda_{0,\mp}^2 \oplus S_{\pm}^2 \oplus \mathbf{1} \oplus S_{0,\mp}^2. \end{aligned}$$

We note that $\Lambda_{0,\mp}^2$ and $S_{0,\mp}^2$ are isomorphic U_{\pm} modules, that $\Lambda_{0,\mp}^2$ is isomorphic to $S_{0,\mp}^2 \otimes \chi$ as a U_{\pm}^* module, and that Λ_{\pm}^2 is not an irreducible U_{\pm} module. We complete our discussion of elementary representation theory with the following diagonalization result (see, for example, the discussion in [7]):

Lemma 2.4. *If ξ is a non-trivial proper \mathcal{U}_{\pm}^* submodule of $\Lambda_{\pm}^2 \oplus \Lambda_{\pm}^2$, then there exists $(a, b) \neq (0, 0)$ so $\xi = \xi(a, b) := \{(a\theta, b\theta)\}_{\theta \in \Lambda_{\pm}^2} \subset \Lambda_{\pm}^2 \oplus \Lambda_{\pm}^2$.*

2.2. The Singer–Thorpe and the Higa decompositions. We now examine the \mathcal{O} -module structure of \mathfrak{R} and \mathfrak{W} . Let

$$S_0^2 := \{\theta \in S^2 : \theta \perp \langle \cdot, \cdot \rangle\} \quad \text{and} \quad \mathfrak{C} := \ker\{\rho\} \cap \mathfrak{R}$$

be the \mathcal{O} modules of trace free symmetric 2-tensors and *Weyl conformal curvature tensors*, respectively. We refer to Singer and Thorpe [33] for the proof of Assertion (1) and to Higa [18, 19] for the proof of Assertion (2) in the following result:

Theorem 2.5. *Let $n \geq 4$.*

- (1) *We may decompose $\mathfrak{R} = \mathbf{1} \oplus S_0^2 \oplus \mathfrak{C}$ as the orthogonal direct sum of irreducible and inequivalent \mathcal{O} modules.*
- (2) *We may decompose $\mathfrak{W} = \mathbf{1} \oplus S_0^2 \oplus \mathfrak{C} \oplus \mathfrak{P}$ as the orthogonal direct sum of irreducible and inequivalent \mathcal{O} modules. Here ρ_a provides an \mathcal{O} module isomorphism from \mathfrak{P} to Λ^2 with the inverse embedding $\Xi : \Lambda^2 \xrightarrow{\sim} \mathfrak{P} \subset \mathfrak{W}$ given by:*

$$\begin{aligned} \Xi(\psi)(x, y, z, w) &:= 2\psi(x, y)\langle z, w \rangle + \psi(x, z)\langle y, w \rangle - \psi(y, z)\langle x, w \rangle \\ &\quad - \psi(x, w)\langle y, z \rangle + \psi(y, w)\langle x, z \rangle. \end{aligned} \tag{2.b}$$

2.3. The Tricerri-Vanhecke decompositions. The following decompositions of \mathfrak{R} and $\mathfrak{K}_{\pm, \mathfrak{R}}$ as \mathcal{U}_- modules was given by Tricerri and Vanhecke [34] in the positive definite setting; they extend easily to the more general context [4, 5]. The decomposition of \mathfrak{W} as a \mathcal{U}_{\pm}^* module then follows from Lemma 2.3 and Theorem 2.5.

Theorem 2.6. *Let $(V, \langle \cdot, \cdot \rangle, J_{\pm})$ be a (para)-Hermitian vector space. We have the following decompositions of \mathfrak{R} , $\mathfrak{K}_{\pm, \mathfrak{R}}$, and \mathfrak{W} as \mathcal{U}_{\pm}^* modules:*

$$\begin{aligned}\mathfrak{R} &= W_{\pm,1} \oplus \dots \oplus W_{\pm,10}, \\ \mathfrak{K}_{\pm, \mathfrak{R}} &= W_{\pm,1} \oplus W_{\pm,2} \oplus W_{\pm,3}, \\ \mathfrak{W} &= W_{\pm,1} \oplus \dots \oplus W_{\pm,13}.\end{aligned}\tag{2.c}$$

If $n = 4$, we omit the modules $\{W_{\pm,5}, W_{\pm,6}, W_{\pm,10}\}$. If $n = 6$, we omit the module $W_{\pm,6}$. The decomposition of Equation (2.c) is then into irreducible \mathcal{U}_{\pm}^* modules. We have \mathcal{U}_{\pm}^* module isomorphisms:

$$\begin{aligned}W_{\pm,1} &\approx W_{\pm,4} \approx \mathbf{1}, & W_{\pm,2} &\approx W_{\pm,5} \approx S_{0, \mp}^2, & W_{\pm,9} &\approx W_{\pm,13} \approx \Lambda_{\pm}^2, \\ W_{\pm,8} &\approx S_{\pm}^2, & W_{\pm,11} &\approx \chi, & W_{\pm,12} &\approx \Lambda_{0, \mp}^2.\end{aligned}\tag{2.d}$$

With exception of the isomorphisms described in Equation (2.d), these are inequivalent \mathcal{U}_{\pm}^* modules. The isomorphism Ψ from Λ_{\pm}^2 to $W_{\pm,9}$ is given by setting

$$\begin{aligned}\Psi(\psi)(x, y, z, w) &:= 2\langle x, J_{\pm}y \rangle \psi(z, J_{\pm}w) + 2\langle z, J_{\pm}w \rangle \psi(x, J_{\pm}y) \\ &\quad + \langle x, J_{\pm}z \rangle \psi(y, J_{\pm}w) + \langle y, J_{\pm}w \rangle \psi(x, J_{\pm}z) \\ &\quad - \langle x, J_{\pm}w \rangle \psi(y, J_{\pm}z) - \langle y, J_{\pm}z \rangle \psi(x, J_{\pm}w).\end{aligned}\tag{2.f}$$

It is worth describing some of these in a bit more detail. Let $\{e_i\}$ be a basis for V . Set $\varepsilon_{ij} := \langle e_i, e_j \rangle$. Define $\rho_{J_{\pm}}(x, y) := \varepsilon^{il} A(e_i, x, J_{\pm}y, J_{\pm}e_l)$. We then have:

$$\begin{aligned}W_{\pm,7} &= \{A \in \mathfrak{R} : A(J_{\pm}x, y, z, w) = A(x, y, J_{\pm}z, w)\}, \\ W_{\pm,3} &= \mathfrak{K}_{\pm, \mathfrak{R}} \cap \ker(\rho), \\ W_{\pm,6} &= \{A \in \mathfrak{R} : J_{\pm}^* A = A\} \cap \{\mathfrak{K}_{\pm, \mathfrak{R}}\}^{\perp} \cap \{W_{\pm,7}\}^{\perp} \cap \ker(\rho \oplus \rho_{J_{\pm}}), \\ W_{\pm,10} &= \{A \in \mathfrak{R} : J_{\pm}^* A = -A\} \cap \ker(\rho \oplus \rho_{J_{\pm}}).\end{aligned}$$

3. THE PROOF OF THEOREM 1.6

If η is an irreducible \mathcal{U}_{\pm}^* module and if ξ is a submodule of $\otimes^4 V^*$, let $n_{\eta}(\xi)$ be the multiplicity with which η appears in the decomposition of ξ given in Lemma 2.2; note that $W_{\pm,4} \approx W_{\pm,1}$ and $W_{\pm,2} \approx W_{\pm,5}$. We apply Theorem 2.6. If η is isomorphic to $W_{i, \pm}$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$, then $n_{\eta}(\Lambda^2) = 0$ so:

$$n_{\eta}(\mathfrak{K}_{\pm, \mathfrak{W}}) = n_{\eta}(\mathfrak{K}_{\pm, \mathfrak{R}}) = \begin{cases} 1 & \text{if } i = 1, 2, 3, 4, 5 \\ 0 & \text{if } i = 6, 7, 8, 10 \end{cases}.$$

Thus only the multiplicities of the representations $\{\chi, \Lambda_{0, \mp}^2, \Lambda_{\pm}^2\}$ are at issue.

3.1. The module $\chi = \Xi(\Omega_{\pm})$ for $m \geq 4$. Let $\{e_i\}$ be an orthonormal basis for V with $J_{\pm}e_{2i-1} = e_{2i}$ and $J_{\pm}e_{2i} = \pm e_{2i-1}$. Let $\varepsilon_{ij} := \langle e_i, e_j \rangle$. We use Equation (2.b) to see:

$$\begin{aligned}\Xi(\Omega_{\pm})(e_1, e_4, e_3, e_1) &= -\langle e_4, J_{\pm}e_3 \rangle \langle e_1, e_1 \rangle = -\varepsilon_{11}\varepsilon_{44}, \\ \mp \Xi(\Omega_{\pm})(e_1, e_4, J_{\pm}e_3, J_{\pm}e_1) &= \pm \langle e_1, J_{\pm}J_{\pm}e_1 \rangle \langle e_4, J_{\pm}e_3 \rangle = \varepsilon_{11}\varepsilon_{44}.\end{aligned}$$

Thus $\Xi(\Omega_{\pm})$ does not satisfy the Kähler identity given in Equation (1.h). Consequently, $n_{\chi}(\mathfrak{K}_{\pm, \mathfrak{W}}) = 0$.

3.2. The module $W_{\pm,12} = \Xi(\Lambda_{0,\mp}^2)$ for $m \geq 6$. Set

$$\psi_{0,\pm} := e^1 \otimes e^2 - e^2 \otimes e^1 - \varepsilon_{11}\varepsilon_{33}\{e^3 \otimes e^4 - e^4 \otimes e^3\}.$$

Clearly $\psi_{0,\pm} \perp \Omega_{\pm}$. Since $J_{\pm}^* \psi_{0,\pm} = \mp \psi_{0,\pm}$, $\psi_{0,\pm} \in \Lambda_{0,\mp}^2$. By Equation (2.b):

$$\begin{aligned} \Xi(\psi_{0,\pm})(e_5, e_1, e_2, e_5) &= -\psi_{0,\pm}(e_1, e_2)\langle e_5, e_5 \rangle = -\varepsilon_{55}, \\ \mp \Xi(\psi_{0,\pm})(e_5, e_1, J_{\pm}e_2, J_{\pm}e_5) &= 0. \end{aligned}$$

Consequently $\Xi(\psi_{0,\pm})$ does not satisfy the Kähler identity and we conclude that $n_{\Lambda_{0,\mp}^2}(\mathfrak{K}_{\pm, \mathfrak{W}}) = 0$ if $m \geq 6$.

3.3. The module $\Lambda_{0,\mp}^2$ if $m = 4$. The argument given above in Section 3.2 does not, of course, pertain if $m = 4$ since we can not examine $\Xi(\psi_{0,\pm})(e_5, e_1, e_2, e_5)$. Let $\eta = \Lambda_{0,\mp}^2$. As noted above, $n_{\eta}(\mathfrak{K}_{\pm, \mathfrak{W}}) \leq 1$. Thus if we can exhibit a non-trivial element of $W_{\pm,12} \cap \mathfrak{K}_{\pm, \mathfrak{W}}$, we will have $n_{\eta}(\mathfrak{K}_{\pm, \mathfrak{W}}) = 1$. We work in the positive definite setting for the moment to simplify the argument. Let

$$\begin{aligned} \psi_{0,+} &:= e^1 \otimes e^2 - e^2 \otimes e^1 - e^3 \otimes e^4 + e^4 \otimes e^3, \\ \langle \cdot, \cdot \rangle &:= e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4. \end{aligned}$$

Decompose $A := \Xi(\psi_{0,+}) = A_1 + A_2 + A_3 + A_4 + A_5$ using the notation of Equation (2.b) where

$$\begin{aligned} A_1(x, y, z, w) &:= 2\psi_{0,+}(x, y)\langle z, w \rangle, & A_2(x, y, z, w) &:= \psi_{0,+}(x, z)\langle y, w \rangle, \\ A_3(x, y, z, w) &:= -\psi_{0,+}(y, z)\langle x, w \rangle, & A_4(x, y, z, w) &:= -\psi_{0,+}(x, w)\langle y, z \rangle, \\ A_5(x, y, z, w) &:= \psi_{0,+}(y, w)\langle x, z \rangle. \end{aligned}$$

As a short hand, we set $e^{ijkl} := e^i \otimes e^j \otimes e^k \otimes e^l$. We may then express:

$$\begin{aligned} A_1 &= 2e^{1211} + 2e^{1222} + 2e^{1233} + 2e^{1244} - 2e^{2111} - 2e^{2122} - 2e^{2133} - 2e^{2144} \\ &\quad - 2e^{3411} - 2e^{3422} - 2e^{3433} - 2e^{3444} + 2e^{4311} + 2e^{4322} + 2e^{4333} + 2e^{4344}, \\ A_2 &= e^{1121} + e^{1222} + e^{1323} + e^{1424} - e^{2111} - e^{2212} - e^{2313} - e^{2414} \\ &\quad - e^{3141} - e^{3242} - e^{3343} - e^{3444} + e^{4131} + e^{4232} + e^{4333} + e^{4434}, \\ A_3 &= -e^{1121} - e^{2122} - e^{3123} - e^{4124} + e^{1211} + e^{2212} + e^{3213} + e^{4214} \\ &\quad + e^{1341} + e^{2342} + e^{3343} + e^{4344} - e^{1431} - e^{2432} - e^{3433} - e^{4434}, \\ A_4 &= -e^{1112} - e^{1222} - e^{1332} - e^{1442} + e^{2111} + e^{2221} + e^{2331} + e^{2441} \\ &\quad + e^{3114} + e^{3224} + e^{3334} + e^{3444} - e^{4113} - e^{4223} - e^{4333} - e^{4443}, \\ A_5 &= e^{1112} + e^{2122} + e^{3132} + e^{4142} - e^{1211} - e^{2221} - e^{3231} - e^{4241} \\ &\quad - e^{1314} - e^{2324} - e^{3334} - e^{4344} + e^{1413} + e^{2423} + e^{3433} + e^{4443}. \end{aligned}$$

We may ignore the terms in A_1 as these belong to \mathfrak{K}_+ . The remaining terms yield a tensor which is anti-symmetric both in the first two and in the last two indices. Thus automatically terms of the form e^{**12} or e^{**34} will belong to \mathfrak{K}_+ and can be ignored. Using the \mathbb{Z}_2 symmetry, we may consider terms e^{ijkl} where $i < j$ and $k < l$. We establish the Kähler identity and show that $n_{\eta}(\mathfrak{K}_{+, \mathfrak{W}}) = 1$ if $m = 4$ in the positive definite setting by examining the following crucial terms:

Term	Coeff.	Term	Coeff.
e^{1323}	$A_2 = 1$	e^{1314}	$A_5 = -1$
e^{1424}	$A_2 = 1$	e^{1413}	$A_5 = 1$
e^{2313}	$A_2 = -1$	e^{2324}	$A_5 = -1$
e^{2414}	$A_2 = -1$	e^{2423}	$A_5 = 1$

We now complexify and let $W := V \otimes_{\mathbb{R}} \mathbb{C}$. Extend $\langle \cdot, \cdot \rangle$, J_- , and A to be complex bilinear, complex linear, and complex multi-linear, respectively. Let:

$$V_{2,2} := \text{Span}_{\mathbb{R}}\{\sqrt{-1}e_1, \sqrt{-1}e_2, e_3, e_4\}.$$

Then $(\langle \cdot, \cdot \rangle, J_-)$ restricts to a pseudo-Hermitian almost complex structure on $V_{2,2}$ of signature $(2, 2)$. Note that

$$\begin{aligned} \text{Re}(A|_{V_{2,2}}) &\in W_{\pm,12}(V_{2,2}) \cap \mathfrak{K}_{-,\mathfrak{W}}(V_{2,2}), \\ \text{Im}(A|_{V_{2,2}}) &\in W_{\pm,12}(V_{2,2}) \cap \mathfrak{K}_{-,\mathfrak{W}}(V_{2,2}). \end{aligned}$$

Since $A|_{V_{2,2}} \neq 0$, at least one of these tensors is non-trivial and the desired conclusion follows for neutral signature $(2, 2)$; a similar argument applied to

$$V_{4,0} := \text{Span}_{\mathbb{R}}\{\sqrt{-1}e_1, \sqrt{-1}e_2, \sqrt{-1}e_3, \sqrt{-1}e_4\}$$

establishes the desired result in signature $(4, 0)$ (which is the negative definite setting). Finally, by considering

$$U_{2,2} := \text{Span}_{\mathbb{R}}\{e_1, \sqrt{-1}e_2, e_3, \sqrt{-1}e_4\}$$

and $J_+ := \sqrt{-1}J_-$, we can construct an example in the para-complex setting.

3.4. The module Λ_{\pm}^2 if $m \geq 6$. Let $\eta = \Lambda_{\pm}^2$. Then $W_{\pm,9} \oplus W_{\pm,13} \approx 2 \cdot \eta$. We adopt the notation of Equation (2.b) and of Equation (2.f). For $(a, b) \neq (0, 0)$, let

$$\xi(a, b) := \text{Range}\{a\Xi + b\Psi\} \subset W_{\pm,9} \oplus W_{\pm,13}.$$

By Lemma 2.4, every non-trivial proper submodule of $W_{\pm,9} \oplus W_{\pm,13}$ is isomorphic to $\xi(a, b)$ for some $(a, b) \neq 0$. We suppose $\xi(a, b) \subset \mathfrak{K}_{\pm,\mathfrak{W}}$ and thus

$$(a\Xi + b\Psi)\psi \in \mathfrak{K}_{\pm,\mathfrak{W}} \quad \text{for all } \psi \in \Lambda_{\pm}^2.$$

Set $\psi_{\pm} := e^1 \otimes e^3 - e^3 \otimes e^1 \pm e^2 \otimes e^4 \mp e^4 \otimes e^2$. Then $J_{\pm}^* \psi_{\pm} = \pm \psi_{\pm}$ so $\psi_{\pm} \in \Lambda_{\pm}^2$. We show that $b = 0$ by checking:

$$\begin{aligned} a\Xi(\psi_{\pm})(e_5, e_6, e_1, e_4) &= 0, \\ \mp a\Xi(\psi_{\pm})(e_5, e_6, J_{\pm}e_1, J_{\pm}e_4) &= 0, \\ b\Psi(\psi_{\pm})(e_5, e_6, e_1, e_4) &= 2b\langle e_5, J_{\pm}e_6 \rangle \psi_{\pm}(e_1, J_{\pm}e_4) = 2b\varepsilon_{55}, \\ \mp b\Psi(\psi_{\pm})(e_5, e_6, J_{\pm}e_1, J_{\pm}e_4) &= \mp 2b\langle e_5, J_{\pm}e_6 \rangle \psi_{\pm}(J_{\pm}e_1, J_{\pm}J_{\pm}e_4) = -2b\varepsilon_{55}. \end{aligned}$$

We show that $a = 0$ and complete the proof of Theorem 1.6 if $m \geq 6$ by checking:

$$\begin{aligned} a\Xi(\psi_{\pm})(e_5, e_1, e_3, e_5) &= -a\psi_{\pm}(e_1, e_3)\langle e_5, e_5 \rangle = -a\varepsilon_{55}, \\ \mp a\Xi(\psi_{\pm})(e_5, e_1, e_4, e_6) &= 0. \end{aligned}$$

3.5. The module Λ_{\pm}^2 if $m = 4$. Again, the argument given in Section 3.4 is not available if $m = 4$ since, for example, we can not examine (e_5, e_1, e_4, e_6) . Let $\eta = \Lambda_{\pm}^2$. Again, we first work in the positive definite setting. Since $n_{\eta}(\mathfrak{K}_{+,\mathfrak{R}}) = 0$, to show $n_{\eta}(\mathfrak{K}_{+,\mathfrak{W}}) = 1$ it suffices to construct a suitable element of $\mathfrak{K}_{+,\mathfrak{W}}$. Let

$$\begin{aligned} \psi_- &:= e^1 \otimes e^3 - e^3 \otimes e^1 - e^2 \otimes e^4 + e^4 \otimes e^2 \in \Lambda_-^2, \\ \langle \cdot, \cdot \rangle &:= e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4. \end{aligned}$$

Adopt the notation of Equation (2.b) to decompose $\Xi(\psi_-) = F + G + H + J + K$ where

$$\begin{aligned} F(x, y, z, w) &:= 2\psi_-(x, y)\langle z, w \rangle, & G(x, y, z, w) &:= \psi_-(x, z)\langle y, w \rangle, \\ H(x, y, z, w) &:= -\psi_-(y, z)\langle x, w \rangle, & J(x, y, z, w) &:= -\psi_-(x, w)\langle y, z \rangle, \\ K(x, y, z, w) &:= \psi_-(y, w)\langle x, z \rangle. \end{aligned}$$

We compute:

$$\begin{aligned} F &= +2e^{1311} + 2e^{1322} + 2e^{1333} + 2e^{1344} - 2e^{3111} - 2e^{3122} - 2e^{3133} - 2e^{3144} \\ &\quad + 2e^{4211} + 2e^{4222} + 2e^{4233} + 2e^{4244} - 2e^{2411} - 2e^{2422} - 2e^{2433} - 2e^{2444}, \end{aligned}$$

$$\begin{aligned}
G &= +e^{1131} + e^{1232} + e^{1333} + e^{1434} - e^{3111} - e^{3212} - e^{3313} - e^{3414} \\
&\quad + e^{4121} + e^{4222} + e^{4323} + e^{4424} - e^{2141} - e^{2242} - e^{2343} - e^{2444}, \\
H &= -e^{1131} - e^{2132} - e^{3133} - e^{4134} + e^{1311} + e^{2312} + e^{3313} + e^{4314} \\
&\quad - e^{1421} - e^{2422} - e^{3423} - e^{4424} + e^{1241} + e^{2242} + e^{3243} + e^{4244}, \\
J &= -e^{1113} - e^{1223} - e^{1333} - e^{1443} + e^{3111} + e^{3221} + e^{3331} + e^{3441} \\
&\quad - e^{4112} - e^{4222} - e^{4332} - e^{4442} + e^{2114} + e^{2224} + e^{2334} + e^{2444}, \\
K &= +e^{1113} + e^{2123} + e^{3133} + e^{4143} - e^{1311} - e^{2321} - e^{3331} - e^{4341} \\
&\quad + e^{1412} + e^{2422} + e^{3432} + e^{4442} - e^{1214} - e^{2224} - e^{3234} - e^{4244}.
\end{aligned}$$

Next we examine the role of Ψ . Set $\tilde{\varepsilon}(x, y) := \langle x, Jy \rangle$ and $\tilde{\psi}_-(x, y) := \psi_-(x, Jy)$. We expand $\Psi(\psi_-) = R + S + T + U + V + W$ where

$$\begin{aligned}
R(x, y, z, w) &:= 2\tilde{\varepsilon}(x, y)\tilde{\psi}_-(z, w), & S(x, y, z, w) &:= 2\tilde{\varepsilon}(z, w)\tilde{\psi}_-(x, y), \\
T(x, y, z, w) &:= \tilde{\varepsilon}(x, z)\tilde{\psi}_-(y, w), & U(x, y, z, w) &:= \tilde{\varepsilon}(y, w)\tilde{\psi}_-(x, z), \\
V(x, y, z, w) &:= -\tilde{\varepsilon}(x, w)\tilde{\psi}_-(y, z), & W(x, y, z, w) &:= -\tilde{\varepsilon}(y, z)\tilde{\psi}_-(x, w).
\end{aligned}$$

We compute:

$$\begin{aligned}
\tilde{\varepsilon} &= -e^1 \otimes e^2 + e^2 \otimes e^1 - e^3 \otimes e^4 + e^4 \otimes e^3, \\
\tilde{\psi}_- &= -e^1 \otimes e^4 + e^4 \otimes e^1 - e^2 \otimes e^3 + e^3 \otimes e^2, \\
R &= -2e^{1241} + 2e^{2141} - 2e^{3441} + 2e^{4341} + 2e^{1214} - 2e^{2114} + 2e^{3414} - 2e^{4314} \\
&\quad - 2e^{1232} + 2e^{2132} - 2e^{3432} + 2e^{4332} + 2e^{1223} - 2e^{2123} + 2e^{3423} - 2e^{4323}, \\
S &= -2e^{4112} + 2e^{4211} - 2e^{4134} + 2e^{4143} + 2e^{1412} - 2e^{1421} + 2e^{1434} - 2e^{1443} \\
&\quad - 2e^{3212} + 2e^{3221} - 2e^{3234} + 2e^{3243} + 2e^{2312} - 2e^{2321} + 2e^{2334} - 2e^{2343}, \\
T &= -e^{1421} + e^{2411} - e^{3441} + e^{4431} + e^{1124} - e^{2114} + e^{3144} - e^{4134} \\
&\quad - e^{1322} + e^{2312} - e^{3342} + e^{4332} + e^{1223} - e^{2213} + e^{3243} - e^{4233}, \\
U &= -e^{4112} + e^{4211} - e^{4314} + e^{4413} + e^{1142} - e^{1241} + e^{1344} - e^{1443} \\
&\quad - e^{3122} + e^{3221} - e^{3324} + e^{3423} + e^{2132} - e^{2231} + e^{2334} - e^{2433}, \\
V &= e^{1412} - e^{2411} + e^{3414} - e^{4413} - e^{1142} + e^{2141} - e^{3144} + e^{4143} \\
&\quad + e^{1322} - e^{2321} + e^{3324} - e^{4323} - e^{1232} + e^{2231} - e^{3234} + e^{4233}, \\
W &= e^{4121} - e^{4211} + e^{4341} - e^{4431} - e^{1124} + e^{1214} - e^{1344} + e^{1434} \\
&\quad + e^{3122} - e^{3212} + e^{3342} - e^{3432} - e^{2123} + e^{2213} - e^{2343} + e^{2433}.
\end{aligned}$$

We may ignore the F and the S terms as these belong to \mathfrak{K}_+ . The remaining terms yield a tensor which is anti-symmetric in the first indices and anti-symmetric in the last indices. Thus automatically things of the form e^{**12} or e^{**34} belong to \mathfrak{K}_+ and don't need to be worried about. Thus the only terms which matter are the following:

Term	Coef	Coef	Coef	Contribution
e^{1223}	$J = -1$	$R = 2$	$T = 1$	$-a + 3b$
e^{1214}	$K = -1$	$R = 2$	$W = 1$	$-a + 3b$
e^{3423}	$H = -1$	$R = 2$	$U = 1$	$-a + 3b$
e^{3414}	$G = -1$	$R = 2$	$V = 1$	$-a + 3b$

Thus we must have $-a + 3b = 0$ so we may take $a = 3$ and $b = 1$. This completes the proof in signature $(0, 4)$; the remaining cases are handled using the same techniques used in Section 3.3.

4. THE PROOF OF THEOREM 1.5

Adopt the notation of Equation (1.g). Fix a bilinear form $\varepsilon = (\varepsilon_{ij})$ on \mathbb{R}^m which is \pm -invariant under J_\pm . Let “ \circ ” denote symmetric tensor product. Let $\theta \in S_\mp^2 \otimes S^2$. We form the germ of a pseudo-Riemannian metric which is \pm -invariant under the action of J_\pm by setting:

$$g = \varepsilon + \theta_{ijkl} x^k x^l dx^i \circ dx^j ;$$

g is a (para)-Hermitian metric on a neighborhood \mathcal{O} of 0 in \mathbb{R}^m . By Theorem 1.4 (2) there is a unique Weyl connection $\nabla = \nabla(\theta)$ so that $(\mathcal{O}, J_\pm, g, \nabla)$ is a (para)-Kähler Weyl manifold. Let $\Theta(\theta) := R^\nabla(0)$; Θ defines an equivariant linear map

$$\Theta : S_\mp^2 \otimes S^2 \rightarrow \mathfrak{K}_{\pm, \mathfrak{W}}.$$

To show that Θ is surjective and complete the proof of Theorem 1.5, we must to show:

$$n_\eta(\text{Range}(\Theta)) = 1 \quad \text{for} \quad \eta \in \{\mathbf{1}, S_{0,\pm}^2, W_{\pm,3}, \Lambda_{0,\mp}^2, \Lambda_\pm^2\}.$$

4.1. The representations $W_{\pm,i}$ for $i = 1, 2, 3$. Let $R^g(0)$ be the curvature of the Levi-Civita connection at the origin. The map $\mathcal{L} : \theta \rightarrow R^g(0)$ is a linear function of θ given by:

$$(\mathcal{L}\theta)(x, y, z, w) := \theta(x, z, y, w) + \theta(y, w, x, z) - \theta(x, w, y, z) - \theta(y, z, x, w).$$

We set $A := \mathcal{L}(\theta)$. Similarly the map $K_\pm : \Theta \rightarrow d\Omega^g$ is a linear map which takes $S_\mp^2(V^*) \otimes S^2(V^*)$ to $\Lambda^3(V^*) \otimes V^*$. It is given by:

$$\{(K_\pm \Theta)(x, y, z)\}(w) := \Theta(x, J_\pm y, z, w) + \Theta(y, J_\pm z, x, w) + \Theta(z, J_\pm x, y, w).$$

This shows that $\ker(K_\pm)$ is invariant under the action of \mathcal{U}_\pm^* . Clearly $\theta \in \ker(K_\pm)$ if and only if g_θ is a Kähler metric. On $\ker(K_\pm)$, we have $\Theta = \mathcal{L}$ since $\phi = 0$. Thus:

$$\mathcal{L} : \ker(K_\pm) \rightarrow W_{1,\pm} \oplus W_{2,\pm} \oplus W_{3\pm}.$$

Take

$$\Theta = \frac{1}{2}(e^1 \otimes e^1 \mp e^2 \otimes e^2) \otimes (e^1 \otimes e^1 + e^2 \otimes e^2)$$

so that the metric has the form

$$g_\Theta = \varepsilon + \frac{1}{2}(u_1^2 + u_2^2)(du_1^2 \mp du_2^2).$$

The metric g_Θ is Kähler since it takes the form $M_2 \times \mathbb{C}$ where M_2 is a Riemann surface. Thus $\Theta \in \ker(K_\pm)$. Furthermore, the only non-zero curvature components of the curvature tensor $A = R^g(0)$ at the origin, up to the usual \mathbb{Z}_2 symmetries, are given by

$$A(e_1, e_2, e_2, e_1) = 1.$$

The symmetric Ricci tensor $\rho_s(x, y) := \frac{1}{2}(\rho(x, y) + \rho(y, x))$ defines a map from $\mathfrak{K}_{\pm, \mathfrak{W}}$ to S^2 . We have

$$\rho_s(e_i, e_j) = \begin{cases} \varepsilon_{22} & \text{if } i = j = 1, \\ \varepsilon_{11} & \text{if } i = j = 2, \\ 0 & \text{otherwise} \end{cases}.$$

Since ρ_s is neither a multiple of $\langle \cdot, \cdot \rangle$ nor is ρ_s perpendicular to $\langle \cdot, \cdot \rangle$, ρ_s has components both in $\mathbf{1}$ and in $S_{0,\mp}^2$. Consequently

$$W_{\pm,1} \oplus W_{\pm,2} \subset \mathcal{L}(K_\pm).$$

Let $S \in S_{\mp}^2$. Following [34], define:

$$\begin{aligned} S_1(x, y, z, w) &:= \langle x, z \rangle S(y, w) + \langle y, w \rangle S(x, z) \\ &\quad - \langle x, w \rangle S(y, z) - \langle y, z \rangle S(x, w) \\ S_2(x, y, z, w) &:= 2\langle x, J_{\pm}y \rangle S(z, J_{\pm}w) + 2\langle z, J_{\pm}w \rangle S(x, J_{\pm}y) \\ &\quad + \langle x, J_{\pm}z \rangle S(y, J_{\pm}w) + \langle y, J_{\pm}w \rangle S(x, J_{\pm}z) \\ &\quad - \langle x, J_{\pm}w \rangle S(y, J_{\pm}z) - \langle y, J_{\pm}z \rangle S(x, J_{\pm}w). \end{aligned}$$

Then the map $\Sigma : S \rightarrow S_1 \mp S_2$ splits ρ_s modulo a suitable normalizing constant.¹ We have:

$$\Sigma(\rho_s)(e_1, e_3, e_3, e_1) = -\varepsilon_{33}\varepsilon_{22}$$

and thus $\Sigma(\rho_s)$ is not a multiple of R so R has a non-zero component in $W_{\pm,3}$ and

$$W_{\pm,3} \subset \mathcal{L}(K_{\pm}).$$

4.2. The representations Λ_{\pm}^2 and $\Lambda_{0,\mp}^2$. The alternating part of the Ricci tensor, ρ_a provides a map from $\mathfrak{K}_{\pm, \mathfrak{W}}$ to Λ^2 . If we can show $\rho_a \Theta$ is a surjective map to $\Lambda_{0,\mp}^2 \oplus \Lambda_{\pm}^2$, it will follow from Lemma 2.2 that $n_{\eta}(\mathfrak{K}_{\pm, \mathfrak{W}}) \geq 1$ which will complete the proof. We have that ϕ is a multiple of $J_{\pm}^* \delta \Omega_{\pm}$ and that $d\phi$ is a multiple of ρ_a . Thus it will suffice to give an example where $dJ_{\pm}^* \delta \Omega_{\pm}$ has components in both $\Lambda_{0,\mp}^2$ and Λ_{\pm}^2 . Suppose $f(x) = x_1 x_3$. Let:

$$ds^2 := \varepsilon_{11} e^{2f(x_1, x_3)} (dx^1 \otimes dx^1 \mp dx^2 \otimes dx^2) + \varepsilon_{22} (dx^3 \otimes dx^3 \mp dx^4 \otimes dx^4).$$

We have [3]:

$$\begin{aligned} (\nabla^g \Omega_{\pm})(\partial_{x_i}, \partial_{x_j}; \partial_{x_k}) &= \frac{1}{2} \{ g(\partial_{x_i}, \partial_{x_k}; J_{\pm} \partial_{x_j}) - g(\partial_{x_j}, \partial_{x_k}; J_{\pm} \partial_{x_i}) \\ &\quad + g(J_{\pm} \partial_{x_i}, \partial_{x_k}; \partial_{x_j}) - g(J_{\pm} \partial_{x_j}, \partial_{x_k}; \partial_{x_i}) \}. \end{aligned}$$

This permits us to compute that:

$$(\nabla^g \Omega_{\pm})(\partial_{x_1}, \partial_{x_3}; \partial_{x_k}) = \begin{cases} \mp \varepsilon_{11} e^{2f} \partial_{x_3} f & \text{if } k = 2 \\ 0 & \text{if } k \neq 2 \end{cases}.$$

The covariant derivative of the Kähler form has the symmetries [3]:

$$\begin{aligned} (\nabla^g \Omega_{\pm})(x, y; z) &= -(\nabla^g \Omega_{\pm})(y, x; z) = \pm (\nabla^g \Omega_{\pm})(J_{\pm} x, J_{\pm} y; z) \\ &= \mp (\nabla^g \Omega_{\pm})(x, J_{\pm} y; J_{\pm} z). \end{aligned}$$

It now follows that the non-zero components of $\nabla^g \Omega_{\pm}$ are given, up to the \mathbb{Z}_2 symmetry in the first components, by:

$$\begin{aligned} (\nabla^g \Omega_{\pm})(\partial_{x_1}, \partial_{x_3}; \partial_{x_2}) &= \mp \varepsilon_{11} e^{2f} \partial_{x_3} f, \\ (\nabla^g \Omega_{\pm})(\partial_{x_1}, \partial_{x_4}; \partial_{x_1}) &= \pm \varepsilon_{11} e^{2f} \partial_{x_3} f, \\ (\nabla^g \Omega_{\pm})(\partial_{x_2}, \partial_{x_4}; \partial_{x_2}) &= -\varepsilon_{11} e^{2f} \partial_{x_3} f, \\ (\nabla^g \Omega_{\pm})(\partial_{x_2}, \partial_{x_3}; \partial_{x_1}) &= \pm \varepsilon_{11} e^{2f} \partial_{x_3} f. \end{aligned}$$

This then implies

$$\begin{aligned} J_{\pm}^* \delta \Omega_{\pm} &= 2 \mp \partial_{x_3} f \cdot dx^3, \\ dJ_{\pm}^* \delta \Omega_{\pm} &= 2 \mp \partial_{x_1} \partial_{x_3} f \cdot dx^1 \wedge dx^3. \end{aligned}$$

This has components in both $\Lambda_{0,\mp}^2$ and in Λ_{\pm}^2 . The desired result now follows.

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¹This result was established in the positive definite setting; it extends easily to the general context

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PG: MATHEMATICS DEPARTMENT, UNIVERSITY OF OREGON, EUGENE OR 97403 USA, E-MAIL: GILKEY@UOREGON.EDU

SN: MATHEMATICAL INSTITUTE, SANU, KNEZ MIHAILOVA 36, P.P. 367, 11001 BELGRADE, SERBIA, EMAIL: STANAN@MI.SANU.AC.RS